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Unveiling problem-solving strategies of pre-service mathematics teachers: A visual and discursive exploration

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Abstract

This study explores how pre-service mathematics teachers (PMTs) in South Africa use visualization and self-discourse to solve mathematical problems. Visualization is known to enhance mathematics learning, while effective communication skills are critical for teaching and learning mathematics, especially in contexts, where the language of instruction may not be the first language of students or teachers. By understanding the visualization techniques and discursive properties employed by PMTs, insights can be gained into how to improve mathematics learning and teaching. The study is informed by the commognitive framework and uses qualitative data from a purposive sample of 10 PMTs who participated in a performance test and semi-structured interviews. The study found that PMTs rely on mental visualization for simpler mathematical problems but use both symbolic and iconic visual mediators for more complicated problems. The use of language to engage in self-discursive activity during problem-solving was found to be key for successful visualization.

Keywords: visualization, mathematical discourse, commognition, problem-solving

INTRODUCTION

Mathematical problem-solving is a crucial aspect of learning and teaching mathematics. In particular, visualization and discourse are two essential components of the problem-solving process (Mudaly, 2021). Visualization techniques assist learners to understand a problem, explore potential solutions, and develop new insights (Samosa et al., 2021). Moreover, effective discourse can enhance communication, promote reasoning, and support the development of a shared understanding of the problem (Albay, 2019).

Despite the importance of visualization and discourse in mathematical problem-solving, there is little established knowledge about how pre-service mathematics teachers (PMTs) utilize visualization and discourse in their problem-solving practices in South Africa and other parts of the world. A lack of understanding of these practices limits the ability of instructors to prepare PMTs with effective strategies and techniques to support their own problem-solving and their future students' learning. The limited knowledge about the use of visualization and discourse by PMTs in mathematical problem-solving obstructs their development of effective teaching strategies (Lee & Lee, 2019; Mahlaba & Mudaly, 2022). Additionally, as language barriers pose a challenge in learning environments, where teachers and learners have different first languages, university instructors should be equipped with expert knowledge to teach PMTs how to use discourse to facilitate communication with students who may not share their first language (Kahiya & Brijlall, 2021).

Visualization of Mathematical Concepts

Espinosa (1997) asserts that visualization is the capacity of the brain to create detailed mental pictures that can be manipulated by the mind by rehearsing various illustrations of the concept, which can then be represented visibly on paper or screen. Visualization is widely recognized as an essential tool for promoting problem-solving skills and enhancing students' understanding of mathematical concepts (Tiwari et al., 2021). Using visual representations such as diagrams, graphs, and models can assist students to connect abstract mathematical concepts to concrete, real-world

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Contribution to the literature

- This study investigates the use of visualization and self-discourse by pre-service mathematics teachers (PMTs) to solve mathematical problems, aiming to improve mathematics learning and teaching in multilingual contexts.
- The research is informed by the commognitive framework, offering a theoretical basis for understanding the visualization techniques and discursive properties employed by PMTs.
- The study highlights the significance of language as a crucial tool for successful visualization, emphasizing the importance of effective communication skills in mathematics education.

situations and develop the spatial reasoning skills crucial for success in mathematics (Tiwari et al., 2021; Yilmaz & Argun, 2018). Visualization also assists students to make connections between mathematical concepts, identify patterns and relationships, and deepen their understanding of mathematics (Rathour et al., 2022).

Mathematical Thinking Through Discourse

Mathematics can be viewed as a science that prioritizes the development of specific thinking skills (Samo & Kartasasmita, 2017). Over the past few decades, there has been an increase in scholarly interest in the intricate processes that support mathematical thought (Stacey, 2007). Alkan and Bukova-Güze (2005) argue that mathematical thinking differs significantly from other types of thinking. It is characterized by learning new information through specialization, generalization, conjecturing, testing, and reasoning, as well as proving (Yildiz, 2016). To deeply engage in mathematical thinking, the thinker must be aware of their thought processes while solving problems, which involves reflecting on their own thinking.

Discourse has been recognized as a powerful tool for promoting mathematical thinking and enhancing students' understanding of mathematical concepts (Hundeland et al., 2020). It is particularly relevant to the education of PMTs as they are expected to engage their learners in discourses that stimulate their mathematical thinking (Toscano et al., 2019). Discourse allows pre-service teachers to articulate their thinking, reflect on their own understanding, and learn from the perspectives of others. The use of mathematical language during discourse can assist pre-service teachers to build connections between mathematical concepts and communicate mathematical ideas more effectively to their learners (Jourdain & Sharma, 2016). However, language barriers between teachers and learners can impact teachers' ability to engage learners in visualization and mathematical problem-solving (Bermejo et al., 2021). This study examined how PMTs employed mathematical discursive properties during problem-solving and how these contributed to their understanding of mathematical concepts.

Van Zoest et al. (2017) point out that some learners participate in mathematical problem-solving that goes beyond the requirements of the curriculum–for example, if they participate in mathematics competitions, such as South African mathematics Olympiad and international mathematics competition (IMC). They contend that teachers should seize opportunities to generate rich teachable moments in the classroom when students pose questions as they grapple with mathematical concepts. In this regard, teachers could probe students to engage deeper with their mathematical thought processes, thus using interpersonal discourse to broaden their intrapersonal discourses, which would positively impact on their visualization abilities. Given the importance of teachers learning to be able to develop and sustain their learners' mathematical thinking, higher education institutions (HEIs) prioritize such engagement during lectures (Celik & Ozdemir, 2020).

Pre-Service Mathematics Teachers' Visualization and Mathematical Thinking Abilities

While visualization has been shown to play a crucial role in enhancing students' understanding of mathematical concepts and promoting problem-solving skills (Osman et al., 2018), research on the visualization abilities of PMTs has produced mixed results. In some global studies, pre-service teachers found it challenging to use visualization effectively in their teaching (Ozsoy, 2018; Shatri & Buza, 2017); in other studies, however, PMTs were able to use visualization effectively to support their own understanding of mathematical concepts and to facilitate student learning (Abate et al., 2022).

Developing mathematical thinking is an essential component of PMTs' professional preparation since they are expected to instill this thinking in their learners at the school level. Research indicates that engaging deeply in mathematical visualization helps students develop more sophisticated mathematical thinking skills, including problem-solving capabilities and the ability to reason mathematically (Yilmaz & Argun, 2018). However, international research on PMTs' mathematical thinking abilities has produced mixed results, with some studies indicating that PMTs struggle to effectively apply mathematical reasoning and problem-solving strategies in their teaching (Kartal et al., 2020; Monson et al., 2020), while others have demonstrated that they can effectively use mathematical thinking skills to support student learning (Tasdan & Kabar, 2022).

This study explored the visualization techniques and discursive properties employed by PMTs in problem-solving and their influence on their problem-solving abilities as well as their pedagogical decisions and practices, with an aim to support the development of effective strategies and techniques for preparing PMTs to teach mathematics.

THEORETICAL APPROACH

The concept of "commognition" in mathematics education was developed by Sfard (2008) in recognition that "interpersonal communication and individual thinking are two facets of the same phenomenon" (p. xvii). In its most basic form, commognition is a cohesive theory of "thinking about thinking within oneself". Commognition relies on four fundamental principles: thinking as individualization of communication; mathematics as a form of discourse; learning mathematics as changing a discourse; and commognitive conflict as a source of mathematical learning. These are explained briefly here.

Thinking as Individualization of Communication

Sfard (2008) argues that human thinking is a form of communication that occurs with oneself that resembles interpersonal communication. The commognitive perspective focuses on thinking as a dialogical process that involves interactions within oneself such as notifying, arguing, questioning, and responding to thoughts. Sfard (2015) emphasizes the inseparability of thought and its expression, whether through verbal or nonverbal means. This framework can be applied to engagement with mathematical problems, where individuals reflect on their own thinking while solving problems.

Mathematics as a Form of Discourse

Sfard (2007) stated that there are certain rules governing both individualized intrapersonal communication and interpersonal communication that individuals may follow unconsciously. Wittgenstein (1953) compares communication to a game with various tools and rules, where individuals may engage in some forms of communication and not others. Communication is related to discourse, which unites some individuals and alienates others based on their interests and ability to follow rules of communication. Mathematical discourse is a form of discourse with its own rules.

Learning Mathematics as Changing a Discourse

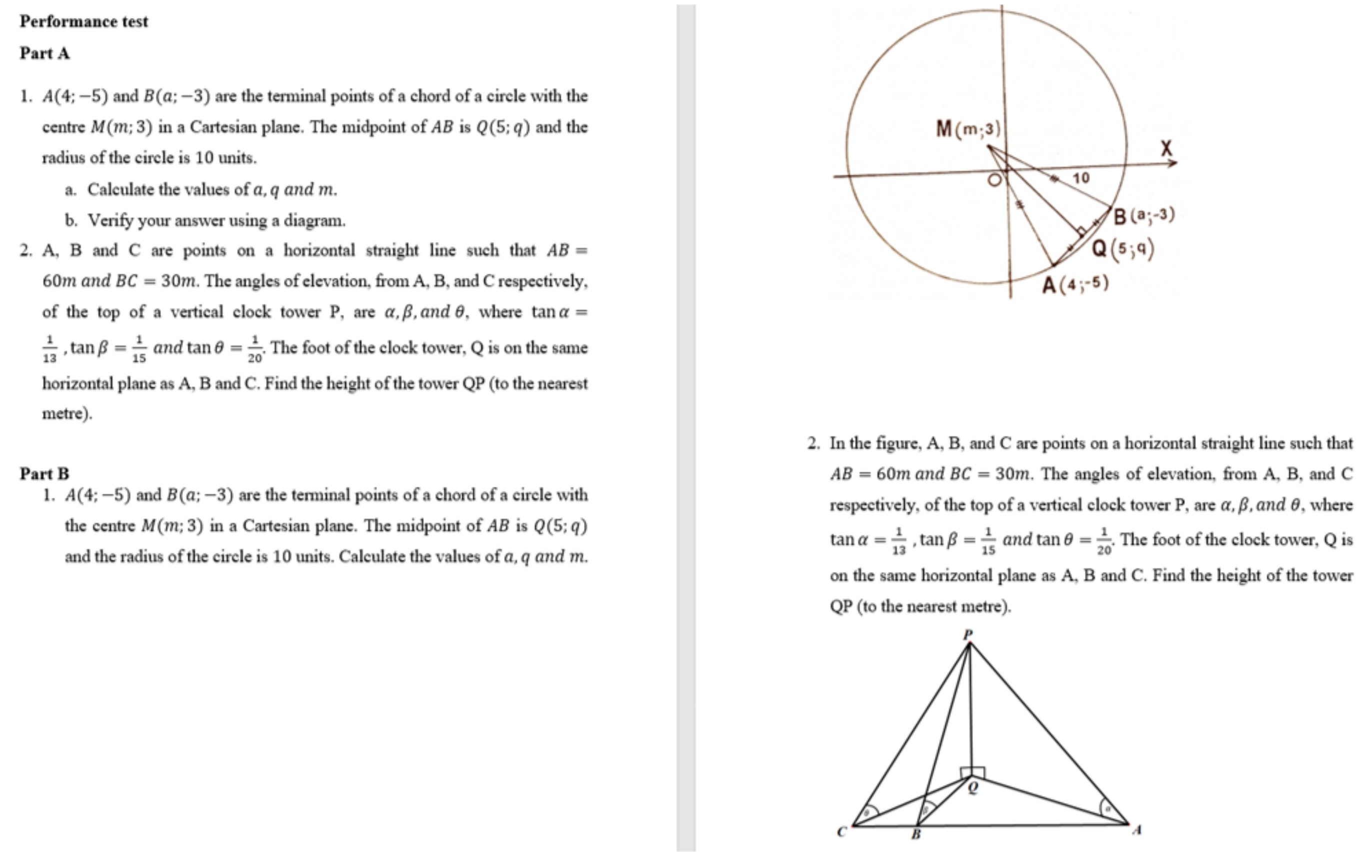
Sfard (2007) maintains that mathematical learning involves transforming existing colloquial discourses, rather than constructing new ones, as mathematical discourse modifies students' existing everyday discourses. Learning to participate in various mathematical discourses in the classroom is therefore fundamentally linked to learning mathematics (Esmonde, 2009). For instance, students can extend and modify their discourse skills to be able to utilize this style of communication during mathematical problem-solving once they have learned concepts linked to algebra or trigonometry. According to Sfard (2007), changes that occur in four discursive characteristics–"the use of words characteristic of the discourse, the use of mediators, endorsed narratives, and routines" (p. 573)–allow for the observation and monitoring of students' discursive development. As a result, it can be inferred that teaching mathematics is a method for altering discourse by witnessing how each of the discursive traits develops as part of an individual's communicational style. However, some mathematics teachers may neglect the discursive processes in which students engage, focusing instead on student mastery of procedures and symbols during mathematics instruction. However, observing and developing the discursive characteristics is crucial for enhancing mathematics learning.

Commognitive Conflict as a Source of Mathematical Learning

Sfard (2007) proposes two types of learning: object-level and meta-level learning. Object-level learning expands existing discourse through vocabulary and the construction of new routines, while meta-level learning alters the meta-rules of discourse. This means that, over time, familiar tasks will begin to be done in a new way. The only way for learners to engage in meta-level learning is through direct interactions with unfamiliar discourses that operate under rules distinct from the discourses they are familiar with (Sfard, 2007). Commognitive conflict arises when discourses do not share criteria for determining whether a specified narrative should be endorsed or not.

Key Commognitive Constructs

Sfard's (2008) commognitive framework employs four features of mathematical discourse: word uses, endorsed narratives, routines, and visual mediators. "Word uses" refers to words used in mathematics discourses; "endorsed narratives" are sequential patterns that refer to mathematical objects, and the relations between these objects, which can be endorsed or denied within mathematics; "routines" are repetitive actions, like drawing graphs; and "visual mediators" refer to objects in mathematics, such as diagrams and mathematical symbols (Sfard, 2008). This study engaged with these four constructs in the first phase of data collection.



Performance test

Part A

1. $A(4;-5)$ and $B(a;-3)$ are the terminal points of a chord of a circle with the centre $M(m;3)$ in a Cartesian plane. The midpoint of AB is $Q(5;q)$ and the radius of the circle is 10 units.
   a. Calculate the values of a, q *and* m.
   b. Verify your answer using a diagram.
2. A, B and C are points on a horizontal straight line such that $AB = 60m$ *and* $BC = 30m$. The angles of elevation, from A, B, and C respectively, of the top of a vertical clock tower P, are $\alpha, \beta,$ *and* θ, where $\tan\alpha = \frac{1}{13}$, $\tan\beta = \frac{1}{15}$ *and* $\tan\theta = \frac{1}{20}$. The foot of the clock tower, Q is on the same horizontal plane as A, B and C. Find the height of the tower QP (to the nearest metre).

Part B

1. $A(4;-5)$ and $B(a;-3)$ are the terminal points of a chord of a circle with the centre $M(m;3)$ in a Cartesian plane. The midpoint of AB is $Q(5;q)$ and the radius of the circle is 10 units. Calculate the values of a, q *and* m.

2. In the figure, A, B, and C are points on a horizontal straight line such that $AB = 60m$ *and* $BC = 30m$. The angles of elevation, from A, B, and C respectively, of the top of a vertical clock tower P, are $\alpha, \beta,$ *and* θ, where $\tan\alpha = \frac{1}{13}$, $\tan\beta = \frac{1}{15}$ *and* $\tan\theta = \frac{1}{20}$. The foot of the clock tower, Q is on the same horizontal plane as A, B and C. Find the height of the tower QP (to the nearest metre).

Figure 1. Performance test screenshot (Source: Authors' own elaboration)

METHODOLOGY

An interpretivism paradigm was employed, with the aim to interpret the phenomenon in its natural context using a deep existential understanding (Lukenchuk, 2013). A multiple case study design was used, with each participant representing a single case. Permission to conduct this study, which originated from a chapter of the first author's PhD research was obtained from the registrar after which ethical clearance was obtained from the university.

PMTs specializing in mathematics education for the further education and training (FET) phase were selected through purposeful sampling at HEI. The researcher had worked as a student tutor and a contract lecturer at the institution for more than five years, affording familiarity with the teaching and learning environment and enabling the immersion in the research context necessary to support an interpretive qualitative exploration (Stake, 2006). 10 students who had achieved the highest grades for a first-year module were selected as the sample cohort. The module included the teaching of geometry and trigonometry content through the use of educational technologies for visualization enhancement.

The study was designed to engage PMTs in individualized and interpersonal mathematical discourses, utilizing visual mediators like drawings, diagrams, graphs, and formulae. Data was collected using a performance test and individual interviews with the 10 participants. The test consisted of two parts: Part A contained two-word problems from analytical geometry and trigonometry of three-dimensional (3D) shapes; participants submitted their responses to part A of the test before receiving part B, which used the same questions but provided diagrams for each question. Interviews were conducted immediately after the performance test while participants could still remember how they had responded to the questions. Subsequent interview questions were on participants' responses to the performance test. The questions on performance test are displayed in **Figure 1**.

A reflexive approach to thematic analysis was used as it provides a flexible theoretical interpretive framework for the examination of qualitative data (Byrne, 2022; Maguire & Delahunt, 2017). Interview recordings were transcribed verbatim and verified by participants. Data generated through performance tests and semi-structured individual interviews were coded deductively and thematized, then interpreted using the commognitive theoretical lenses.

FINDINGS AND DISCUSSION

The study observed three of Sfard's (2008) principles of commognition in operation; the fourth tenet–commognitive conflict as a source of mathematical learning–was not observed. The realization of mathematics as a form of discourse was apparent during the interpretation of interview responses, which were based on the performance test responses.

The four discursive components identified in Sfard's (2008) work–visual mediators, endorsed narrative, routines, and word use–were found. From these, four themes emerged from participants' responses to the performance test and their interview narratives.

Visual Mediators

Iconic (graphic and geometric) and symbolic (formulas and symbols) visual mediators were observed in participants' responses. **Figure 2** presents PMT2's iconic visual mediator to question 1 (**Figure 1**) of part A.

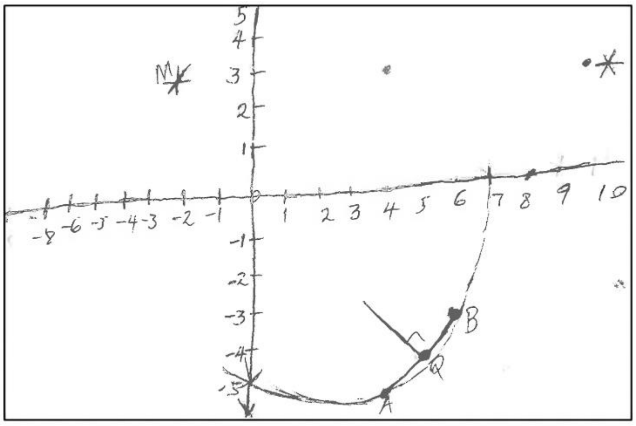


Figure 2. PMT2's visual mediator for question 1 of part A (Source: Authors' own elaboration)

This iconic visual mediator demonstrates how PMT2 visually interpreted the problem. While the sketch is inaccurate, the labels on the diagram match the problem statement and indicate that he comprehended the problem. During the semi-structured interview, PMT2 explained that whenever he comes across a word problem in mathematics, his first step in order to solve it is to create a concrete visual depiction of the problem on paper:

> I'd say, scribbling around on the question paper and trying to sketch something visually is the way to get away with these types of questions; just reading the statement only is not enough to see what you need to do (PMT2).

PMT2 was able to make a concrete visual representation of the problem by scribbling and sketching. Supardi et al. (2021) claim that students frequently use graphic visual mediators when solving word problems in mathematics. Similar visualization techniques were used successfully for the first problem in part A by several other participants. Two of them described their use of visualization as follows:

> I always try to draw it out if I can … So, when I saw the coordinates … I thought if I can just plot it on the Cartesian plane–so I can see how the circle could possibly look like, or where the chord was … (PMT1).

> The strategies to use to solve these types of problems is to sketch some diagrams that depict what you are internalizing as you read, to see what you want to solve and whether the answer makes sense on your drawing … (PMT3).

These participants indicated that they found it easier to understand mathematical word problems using a visual sketch. For PMT1, it was crucial to visualize how the circle appeared on the Cartesian plane given the circle's center and other defining parameters. **Figure 3** illustrates the iconic visual mediator PMT1 used to determine, where the circle was on the Cartesian plane as well as other relevant information.

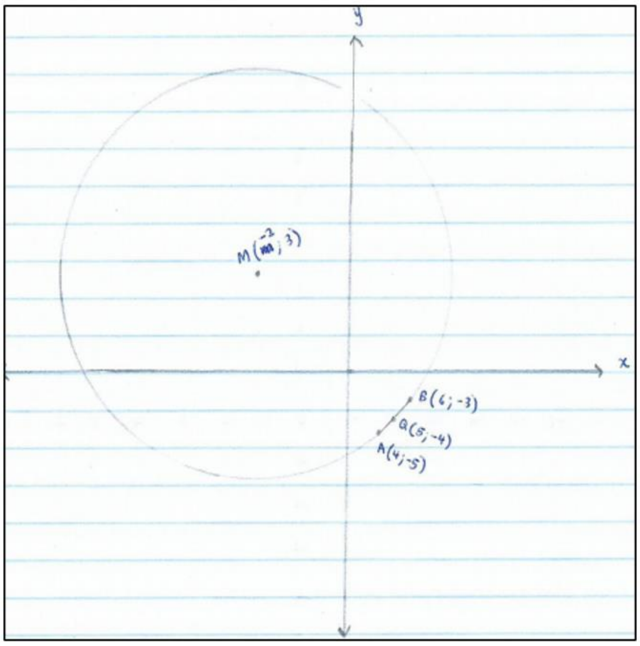


Figure 3. PMT1's visual mediator for question 1 of part A (Source: Authors' own elaboration)

As was found with PMT3, it was important for PMT1 to see the problems as drawings during the internalization and solution-making processes. The internalization process he describes is consonant with Sfard's (2008) tenet of thinking as individualization of communication, where the student engaged in a dialogical process during problem-solving. Additionally, in order to visualize mathematical problems, Arcavi (2003) asserts that "seeing" is a crucial component. PMT3 created a visual depiction of the problem statement as he worked to solve it and figure out the values of the unknowns. As PMT3 worked through the problem and established the values of the unknowns, a visual representation of the problem statement emerged. He explained his iconic visual mediator (**Figure 4**) by stating that the circle that appeared on the Cartesian plane was the final sketch he created; the other three circles floating around the axis were earlier approximations of his visualization as he first read the statement.

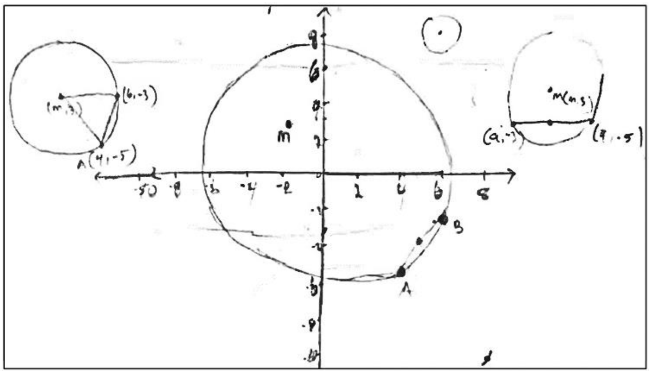


Figure 4. PMT3's visual mediator for question 1 of part A (Source: Authors' own elaboration)

PMT3's description of his visualization processes resonates with Mudaly and Rampersad's (2010) model of the visualization processes. They demonstrate that knowledge is internalized and externalized when a mental representation of meaning is produced by "reflection, interaction with the new stimuli, and other given data" (Mudaly & Reddy, 2016, p. 181). PMT3 used the internalized new knowledge to guide what he added to his visual mediator as it eventually developed into the final image. In addition, PMT4 highlighted, in his narrative, that using sketches to represent the information from a problem statement assisted in solving problems. He stated that

> It is to first note every piece of information they are giving you from the statement of the problem, that is strategy number 1 … then sketch the given information diagrammatically and try to add more information into your sketch as you read the statement going back and forth until it makes sense (PMT4).

PMT4 and PMT5's approaches were similar to PMT3's, as they all developed their iconic visual mediators progressively as they acquired additional information from the problem statement and the steps of their solution as they calculated the values of other unknown variables.

Figure 5 illustrates the iconic visual mediators that PMT4 used to visualize the problem. His narrative also concurred with Mudaly and Rampersad's (2010) concept of visualization processes that are rooted within internalization and externalization activity. Additionally, PMT4's writing down of information diagrammatically and adding information onto the diagram to make sense of the question correlates with several other findings on the interpretation of how students comprehend questions (Diezmann, 2000; Mudaly, 2021; Mudaly & Reddy, 2016). Diezmann (2000) explains that writing information down in a diagram (i.e., creating an iconic visual mediator) is a process of translation that involves decoding linguistic information and encoding visual information in a spiral, iterative manner–as Mudaly (2021) outlines. These

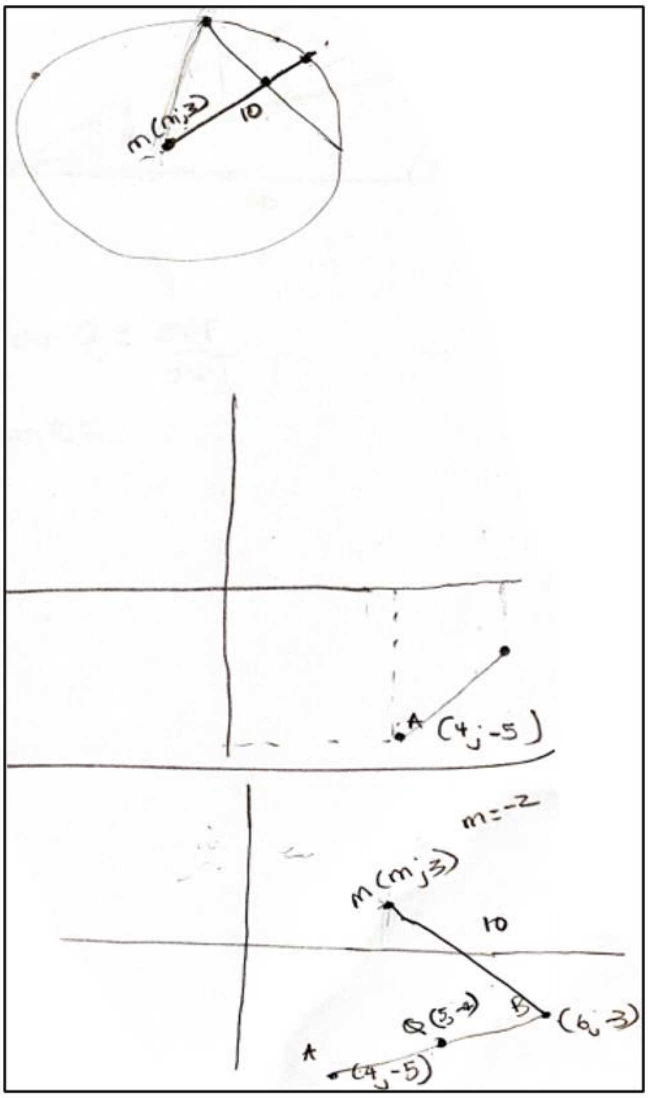


Figure 5. PMT4's visual mediator for question 1 of part A (Source: Authors' own elaboration)

interpretations of the visualization of mathematical problems demonstrate that process of individualization, a fundamental aspect of the commognitive framework, occurred.

Similarly, PMT5's iconic visual mediators, depicted in **Figure 6**, demonstrate how the visualization process progressed–first as she initially internalized the problem statement, and then as she solved the problem.

It is apparent that while these iconic visual mediators were not mathematically accurate, they accurately reflected the individual representations of the participants' interpretation and internalized mental images of these problems.

Figure 7 represents a visual mediator PMT6 sketched to verify her solution for question 1a (**Figure 1**). By looking at her diagram, she realized that her solution was incorrect. However, had there not been a follow-up question requiring her to verify her solution using a diagram, she would have assumed that her calculations were correct. The sketch of her diagram using the calculated unknowns revealed that she was incorrect,

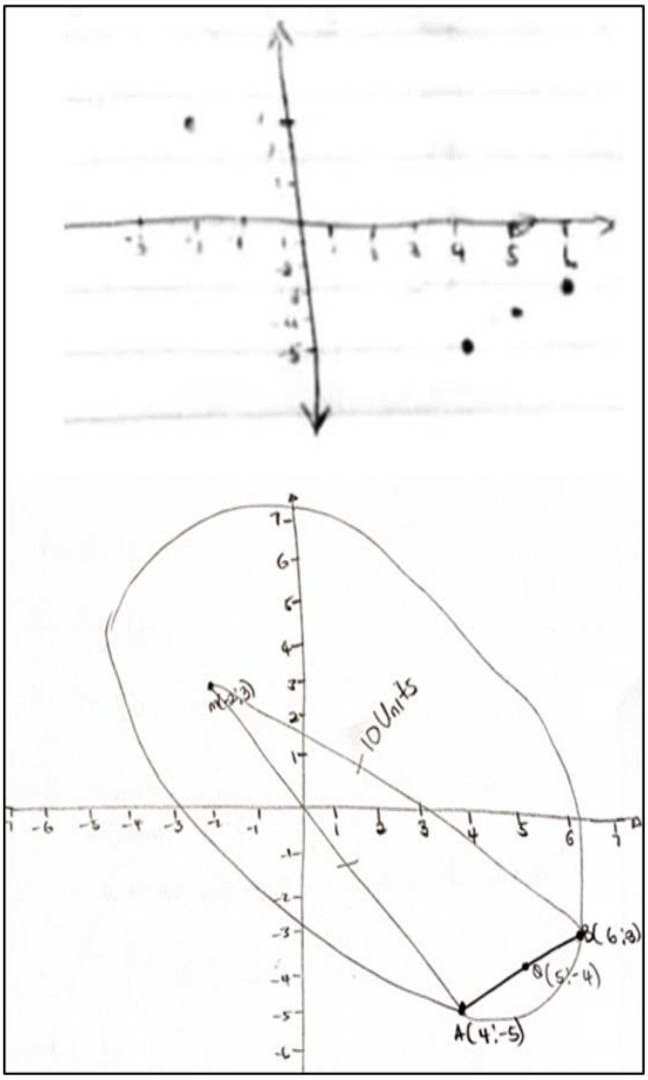


Figure 6. PMT5's visual mediator for question 1 of part A (Source: Authors' own elaboration)

since line segment MQ was supposed to be perpendicular to the chord AB, which was not represented accurately in the diagram indicated. She stated that

> … when I had to verify my answer using a diagram, my values for a, m, and q were incorrect because they did not come out perfectly as a circle with other details corresponding like a chord and perpendicular bisector (PMT6).

Her narrative demonstrates that blind manipulation of symbolic visual mediators and routines may not always yield accurate results. In this regard, Vale and Barbosa (2018) assert that the use of visual methods during mathematical problem-solving is vital, as without them the chances are greater that students will attempt to solve problems without adequate understanding.

All the participants' narratives revealed that the visualization strategies they employed assisted them in verifying their solutions. In addition to the example of PMT6 being able to identify that there was a problem with her calculations after sketching her solution, other participants were able to verify that their solutions were correct using iconic visual mediators. These results align with those of other studies (de Koning et al., 2022; Pantziara et al., 2009) that have established that the significance of diagrams in solution verification for mathematical problems has long been acknowledged by the mathematics education community.

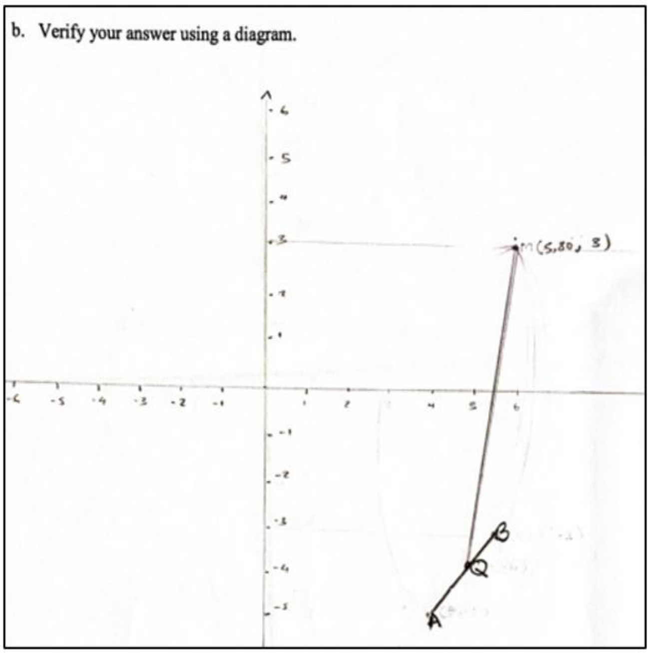


Figure 7. PMT6's visual mediator for question 1 of part A (Source: Authors' own elaboration)

PMT7 and PMT8, however, shared a slightly different approach in terms of their visualization strategies:

> I create mental images without even drawing them down on paper, I visualize them in my head, especially when it is an easy problem like number 1 … (PMT7).
>
> For analytical geometry, every time–before I even begin solving the problem–in my mind I sketch the Cartesian plane. Then I try to plot the coordinates as they appear when I read the statement (PMT8).

This suggests that, depending on the difficulty of a mathematics problem, mental visualization can be used to solve the problem without relying on iconic visual mediators. This was demonstrated in an empirical study conducted by Mudaly (2021), which found that students were able to organize data and develop strategies for solving mathematics problems using only internally visualized diagrams. This approach requires consciously reflecting on one's own thinking during problem-solving, which Sfard (2007) refers to as "intrapersonal communication". This study found that pre-service teachers used iconic visual mediators only for verification after solving the problem, as seen in the verification diagrams of PMT7 and PMT8 in **Figure 8**.

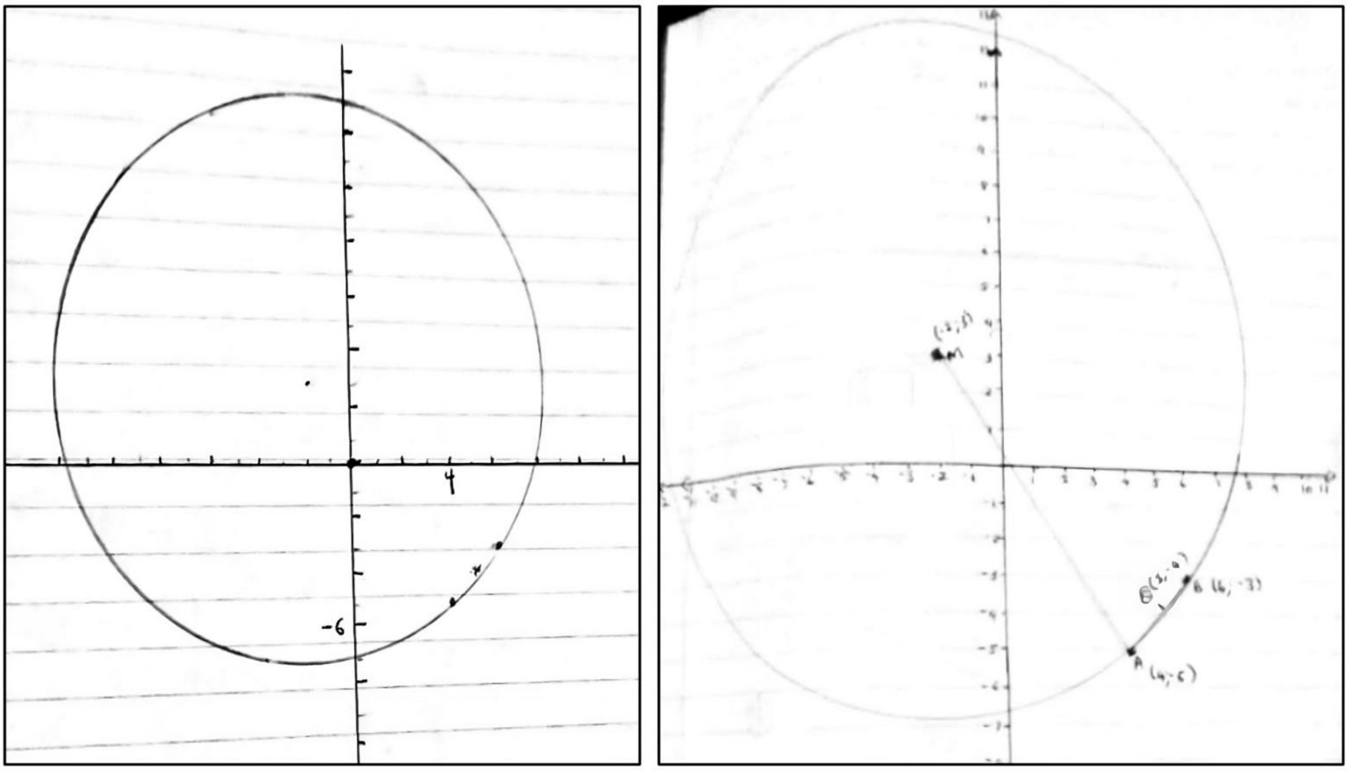


Figure 8. PMT7's (left) and PMT8's (right) visual mediators for question 1 of part A (Source: Authors' own elaboration)

PMT9 had a similar response, as he also relied on his mental visual images while solving the first problem and only made a visual sketch on paper for verification purposes. He maintained:

> I was able to solve number 1 without first sketching a diagram on the paper, it was easy to visualize it … (PMT9).

Figure 9 depicts PMT9's iconic visual mediator illustrating the radius perpendicular to a chord on circle center M as a way to verify his solution to the first question.

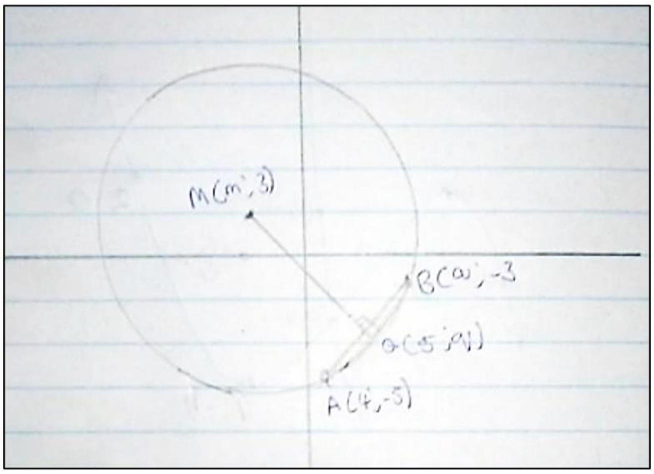


Figure 9. PMT9's visual mediator for question 1 of part A (Source: Authors' own elaboration)

In part A of the test, pre-service participants used visual mediators such as graphic representations of a circle, line segments indicating the chord and its perpendicular bisector, and symbolic visual mediators such as the equation of a circle, midpoint, distance, and slope formulae to solve and verify their responses to the first question. These symbolic visual mediators were essential to solving the question; they are discussed in the following subsection on mathematical word use and routines.

The participants were given a 3D trigonometry problem that required similar visualization skills to the previous question but was more challenging. Only two out of 10 participants produced a correct visual image. The eight participants who did not arrive at the correct answer tried to visualize the problem in 2D format, indicating a limitation in their ability to visualize higher-order mathematics problems.

During individual interviews, these participants explained that they had had difficulty visualizing the problem, which had prevented them from solving it. This highlighted the participants' struggles with visualization in mathematical problem-solving, although the exercise did not focus on acquiring the correct answer. PMT8 described his attempt to solve the problem:

> For the second one, I tried to sketch my diagram to have a visual image and I tried to calculate. But then I ended up having more unknowns–like two or more in one equation–which was then confusing. Also, with my angles, I could not tell as to how I was going to manipulate them accordingly to help me get something useful. So, I can also say that it was hard for me to have a clear

> diagram for number 2; that's why I ended up just putting ABC that way on the straight line (PMT8).

PMT8's narrative confirmed that he was unable to visualize the second problem, as is evident in **Figure 10**.

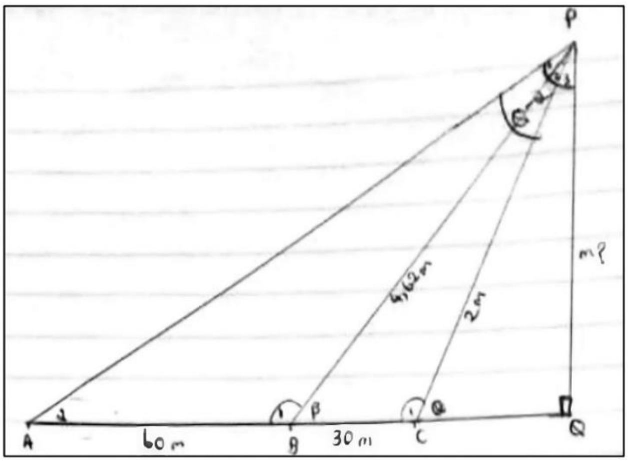


Figure 10. PMT8's visual mediator for question 2 of part A (Source: Authors' own elaboration)

A study conducted by Osman et al. (2018) highlights the significance of visualization in mathematical problem-solving and how it reflects a student's level of understanding and imagination. In this study, participants who failed to visualize the problem had difficulty solving it, indicating a shallow understanding of the problem. PMT8 and PMT3 were unable to visualize the footing of the tower at point Q, despite identifying the height of the tower PQ and the angles of elevation from points A, B, and C. PMT3 explained his thought process, as follows (**Figure 11**):

> … I honestly thought they were talking about the 2D problem. That's the only thing I could think of until I saw part B–then I realized I missed the point of the question there and my sketches were not correct (PMT3).

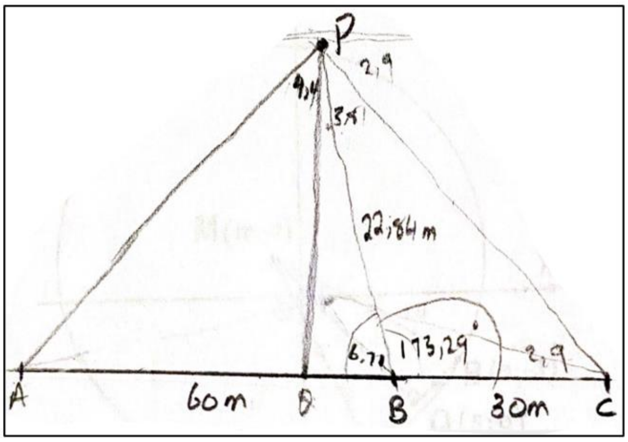


Figure 11. PMT3's visual mediator for question 2 of part A (Source: Authors' own elaboration)

The study found that PMT3 and PMT2 encountered difficulties in solving the 3D trigonometry problem due to limited visual-spatial thinking and incorrect assumptions about the problem's dimensions. PMT3 incorrectly assumed the problem was 2D, while PMT2 had difficulty finding the foot of the tower mentioned in the problem statement. Visual-spatial thinking is crucial in approaching mathematical problems, as it plays an important role in problem-solving (Hawes & Ansari, 2020). **Figure 12** represents the geometric visual mediators PMT2 sketched to visualize this problem.

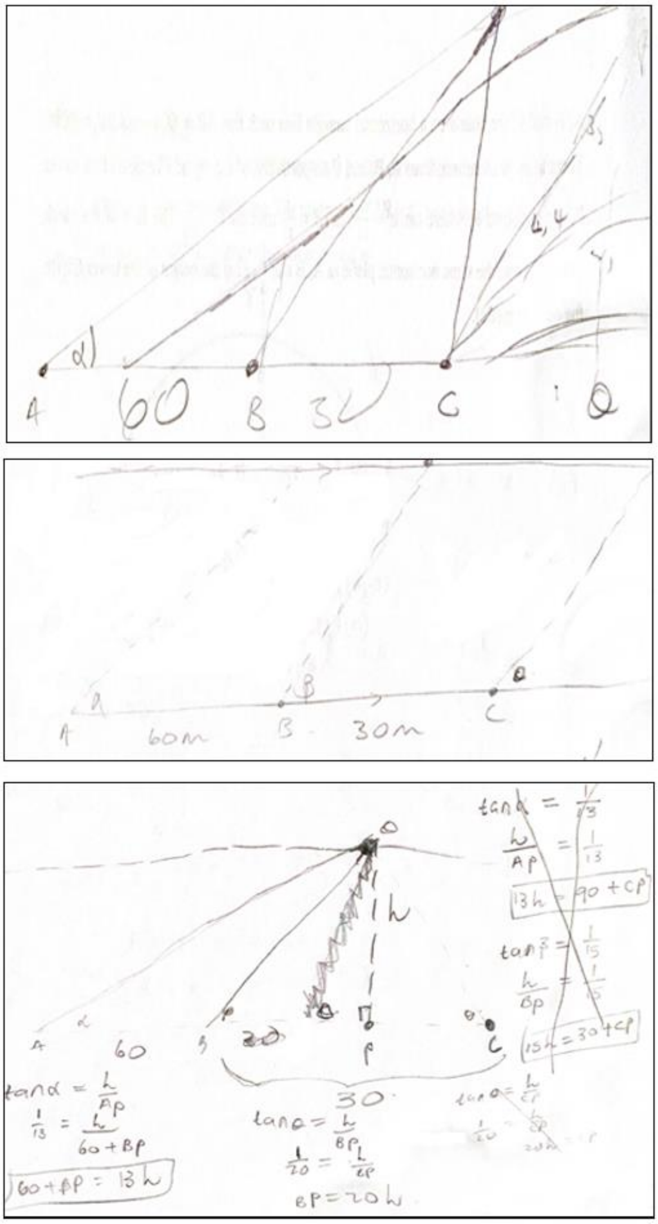


Figure 12. PMT2's visual mediator for question 2 of part A (Source: Authors' own elaboration)

He clearly did not visualize this problem correctly as he plotted Q in each of the three draft sketches in a position that prevented the creation of a 3D representation. He explained during the interview:

> Number 2 was very tricky, because if you are trying to visualize the diagrams and once you have the wrong idea in your head then you will be out in terms of getting the correct answer. Wrong interpretation will result in an incorrect visual

> image and wrong solution based on the misinterpretation of the statement (PMT2).

PMT2 acknowledged that visualization is critical in solving mathematical problems and recognized his own misinterpretation of the problem statement as the reason for his incorrect visual understanding of the problem. This suggests that he understood and acknowledged the importance of visualization in mathematical problem-solving, which could have a positive impact on his teaching as a mathematics teacher. He explained his process while trying to solve the problem further:

> … when I first read this question I realized that I had to do some drawing since I saw a statement talking about the angle of elevation … I thought that was all happening in 2D, which is why everything was just incorrect … when they said you have ABCQ on the same plane I wrongly assumed that they were saying ABCQ is on the same straight line–as you can see on my first sketch (PMT2).

A similar misinterpretation was noted in PMT5's geometric visual mediator, shown in **Figure 13**. She explained:

> … I could not solve it fully, and I do not feel comfortable explaining it because I feel like I'm dismally wrong. I was just guessing … from the ratios that were given of tan. I was guessing, because I did not want to leave a blank space; I could not really solve it. I only reasoned about the ratios … (PMT5).

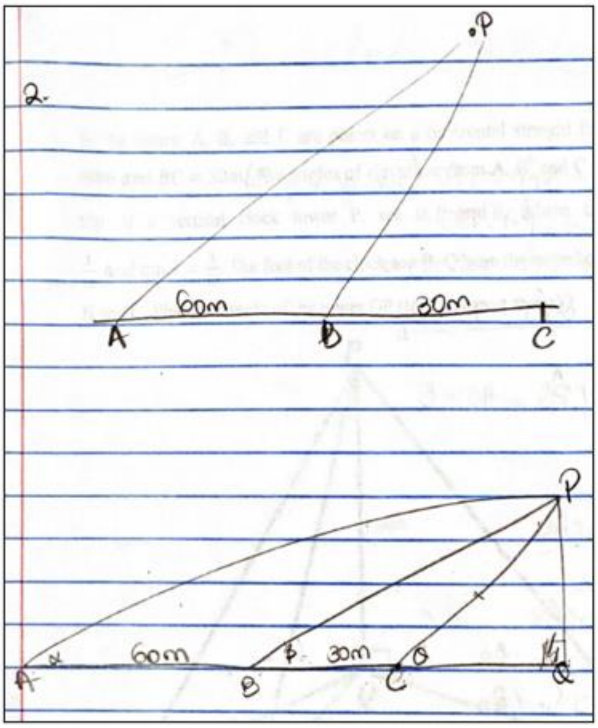


Figure 13. PMT5's visual mediator for question 2 of part A (Source: Authors' own elaboration)

Her description and the sketch, which is depicted in **Figure 13**, illustrate that she did not comprehend the problem statement and was unable to visualize it. This implies further that failing to comprehend the mathematical problem led to failure to visualize it, and consequently failure to solve it.

PMT6, too, found the second problem in part A challenging, as she found it difficult to visualize and to fully comprehend (**Figure 14**):

> … I had two diagrams there. The first one is incomplete. I realized that I was misinterpreting some details from the statement, then I left it. Then I reread the info to come up with the second one–which I was still not sure of its accuracy. To be honest, I could not solve number 2 … I tried to attempt it, but I ended up scratching it out because

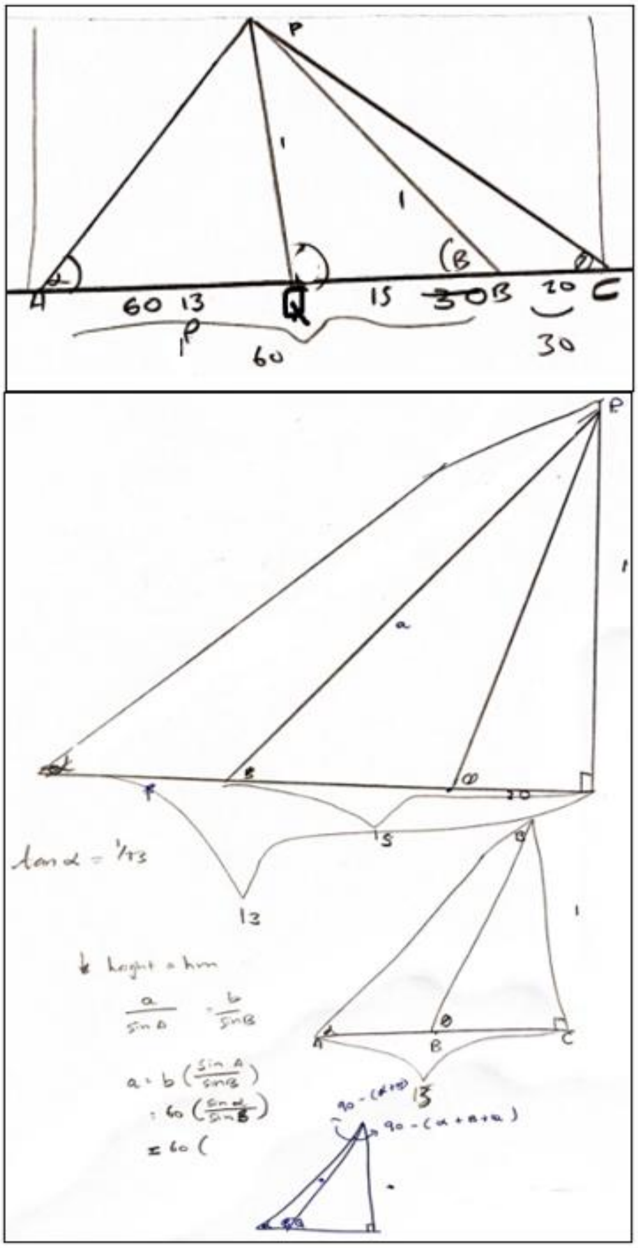


Figure 14. PMT6's visual mediator for question 2 of part A (Source: Authors' own elaboration)

> I could tell that I was not doing it correctly. I did not think that the problem was in 3D … (PMT6).

Moleko (2021) maintains that reading mathematical word problems without full comprehension can make it difficult to reason and create useful mental images or geometric visual aids for problem-solving. In PMT6's case, her limited ability to visualize, combined with inadequate reading comprehension, led to difficulty in solving the second question in part A of the performance test.

Few participants, including PMT4, were able to recognize the second problem in part A as being in 3D on their first attempt. Like other participants, PMT4 was unable to establish the tower PQ's length. His visual mediator for this question is included in **Figure 15**, along with the two steps he jotted down below **Figure 15**.

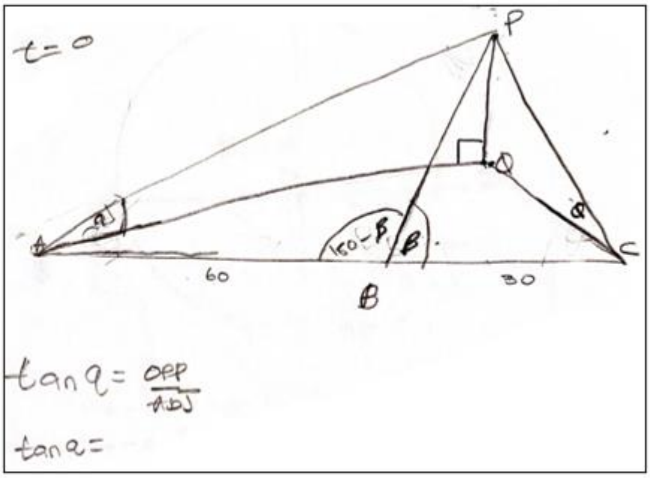


Figure 15. PMT4's visual mediator for question 2 of part A (Source: Authors' own elaboration)

PMT4 had difficulty visualizing the line segment from B to the foot of the tower Q on his geometric visual mediator, resulting in an inaccurate angle of elevation β. He also could not use the information provided and his visual sketch effectively to find the height of the tower.

The challenges experienced by the participants demonstrate two pertinent issues. Firstly, they lacked basic trigonometry knowledge. The level of difficulty of the question would be considered advanced for secondary school learners but should be of average difficulty for third-year university mathematics students. Secondly, despite being third-year mathematics education specialist students, they did not possess sound trigonometry knowledge beyond the level of secondary school content.

PMT4 identified the language used in the question as increasing the level of difficulty of the question for him:

> Part A questions were challenging because you have to visualize the diagram on your own using the information you are given and sometimes it is not clear enough–or, should I say, the type of English they use is a bit tricky. You have to really understand the entire sentence to know what they are saying. For example, about the angle of elevation or depression … it was not easy to put everything together … (PMT4).

The response of PMT4 indicates that language proficiency can have an impact on the understanding of mathematics problems. Studies (Chitera et al., 2016; Robertson & Graven, 2020a, 2020b) have affirmed that second-language English-speaking mathematics students may struggle with word problems, where language proficiency is critical. Although English is the medium of instruction at the institution, where the participants were studying, PMT4's difficulty understanding the problem indicates that language barriers can still be a challenge for third-year mathematics students. Proficiency in English, reading skills, and understanding mathematical terminology are important for success in visualizing mathematics word problems (Moleko, 2021).

The only participant to visualize this problem accurately on the first attempt was PMT1. PMT1's spatial-visual thinking as he worked through this problem is represented by the geometric visual mediator in **Figure 16**. PMT1's sketch was accurate and correct; however, he made one assumption that was not stated in the problem statement, that $\widehat{CQA} = 90°$, for the problem to be solvable according to his interpretation.

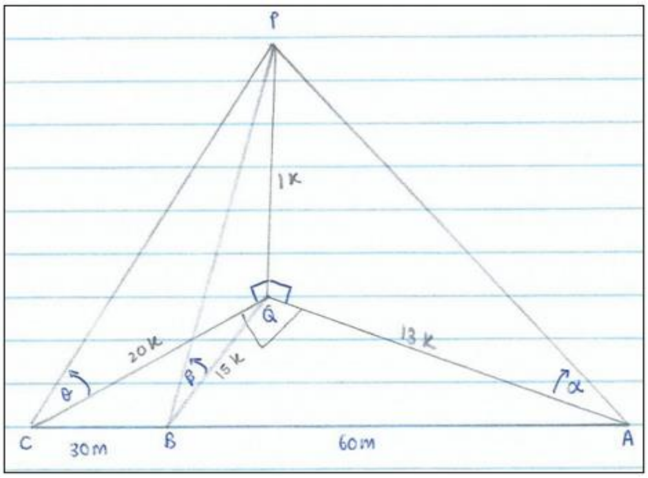


Figure 16. PMT1's visual mediator for question 2 of part A (Source: Authors' own elaboration)

Word Use, Routines, & Endorsed Narratives Employed by PMTs

The study identified multiple instances of participants using mathematical words, endorsed narratives, and routines in their solutions. These observations were gathered from the participants' completed performance tests and from the interviews during which they were requested to describe their individualized mathematical discourses. PMT5's answer appears in **Figure 17**, where symbolic visual mediators such as the equation of a circle, midpoint, distance formula, and several arithmetic operators are evident.

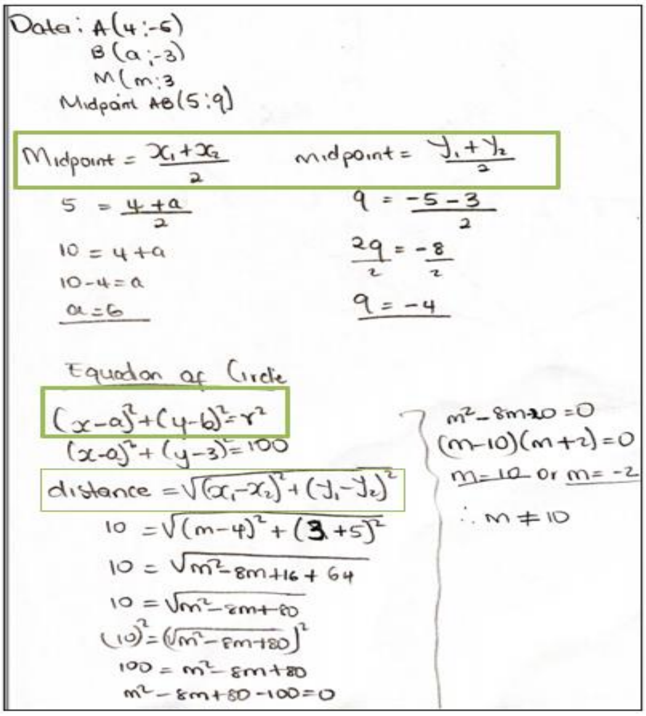


Figure 17. Symbolic (squared) visual mediators from PMT5's solution (Source: Authors' own elaboration)

PMT5 described her approach to solving the problem:

> After reading the statement word by word and sentence by sentence, I was picking up those important things to help me solve the problem … So, I realized that using a midpoint formula right away was a better option, looking at the information that was given ... Midpoint was the first and most important keyword that was giving a direction. So, for the center, I first needed to know the equation of the circle, but since the radius was already given, I substituted everything there. Then I solved for m and, according to my diagram [**Figure 17**], it was clear that m=-2 and not 10 (PMT5).

The participant's knowledge of mathematical concepts such as midpoint, center, and radius was essential for solving the problem. Reading the problem statement carefully assisted her to identify vital information that led to identifying key symbolic visual mediators. This aligns with Moleko's (2021) study, which found that mathematics teachers use the strategy of highlighting keywords to teach problem-solving and visualization skills.

The responses of the other nine participants showed use of mathematical word uses and symbolic visual mediators that was similar to that demonstrated by PMT5; for this reason, they are not presented in detail. However, PMT6's use of the word "distance" was different from that of other participants, as she did not use any formulas in her initial solution. **Figure 18** depicts PMT6's response to the first question, where she used her understanding of the word midpoint and applied an endorsed mathematical narrative.

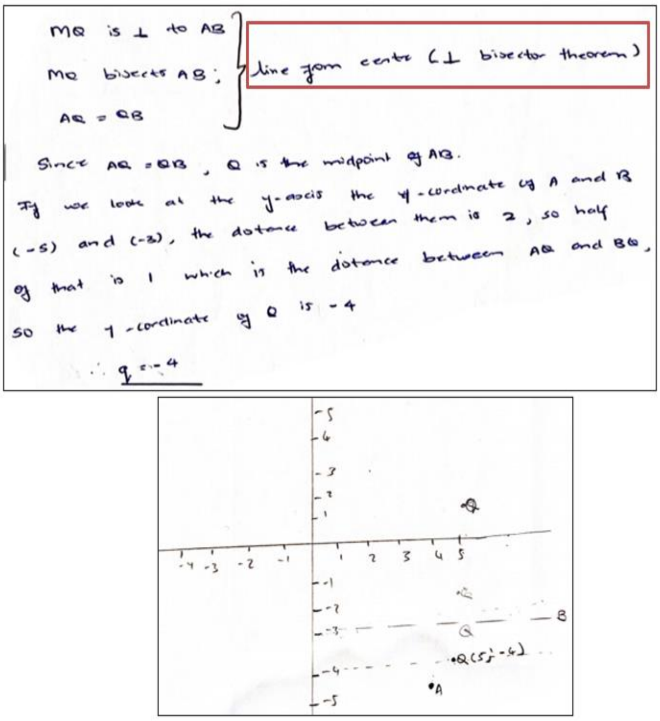


Figure 18. PMT6's solution to number 1 of part A (endorsed narrative [squared]) (Source: Authors' own elaboration)

PMT6 utilized an endorsed mathematical narrative that states that "a line drawn from the center of a circle to a midpoint of a chord is perpendicular to the chord" to assist her to visualize and solve the problem. This approach is illustrated in **Figure 18**. Using this technique, she could visualize the y-axis of a Cartesian plane, where if the endpoints of a line segment AB (a chord) were -5 and -3, respectively, then the center of that line segment should have an ordinate of -4. She described her thought process:

> In calculating Q, I used my diagram, which assisted me to recognize a better and quicker method of getting the unknown. I used the fact that, if they say Q is a midpoint of AB, then the y coordinate of Q could be calculated by saying: if the distance between A and B is 2, then it means the distance from A to Q is 1 and from B to Q is also 1, given the y coordinate of B being -3 and y coordinate of A being -5. Then I saw that the y coordinate of Q had to be -4 since the distance was 1. So, I visualized the coordinates and distances on the Cartesian plane, no formula … (PMT6).

PMT6 solved the problem using mental imagery without relying on specific formulae. She used a graphic visual mediator to obtain the required value and applied the concept of distance in the context of coordinate geometry. Her approach was effective, and her answer was similar to those of participants who had used algebraic techniques. These findings align with Mudaly's (2021) study, emphasizing the importance of encouraging students to use mental imagery during problem-solving.

In addition, a substantiation routine, "solution verification"–which is defined by Gavilán-Izquierdo and Gallego-Sánchez (2021) as a routine for checking results through the use of geometric definitions–was observed in participants' responses. PMT4 described his use of solution verification, as follows:

> … so I used the distance formula, and I substituted the length of the radius and the other unknowns, then I solved it to obtain m=14 and m=-2. Then I went to my diagram to ask myself that, if *m* was equal to 14, is it corresponding with the given information, which stated that MQ is a perpendicular bisector of AB, since M was given to be a center? Then I realized that 14 is too far for it to make sense and to correspond with the given info … (PMT4).

PMT2 used his conceptual understanding of geometry to arrive at a final value for the abscissa of the center. He understood that for a line segment to be a perpendicular bisector to another line segment on a Cartesian plane, it must slant in a particular direction that satisfies the definition of a perpendicular bisector. Since the problem statement in the study provided an ordinate of the center, they had to use analytical geometric reasoning to verify their result if their method yielded two values for the unknown. PMT2's justification also included a similar substantiation routine with a slightly more detailed rationale:

> I substituted everything into the formula of a circle, then I arrived at m=10 or m=-2. Then I had to choose one answer … I tried to plot the coordinates according to these values, where I realized the value that was making sense is -2. Simple, because if I have a chord, these have to be radius perpendicular to the chord and it has to cut it at the midpoint. Then I saw that it has to be on this side [pointing at his diagram] and it cannot be perpendicular coming from 10; then I chose m=-2 (PMT2).

PMT2's justification relied heavily on visualizing the problem, which aided him in recalling specific axioms through intrapersonal communication based on his visual images. Rif'at (2018) argues that visualization in mathematical problem-solving is not only a thinking tool but also an interconnected sequence of reasoning that leads to the development of formal analytical abilities. PMT2's justification involved the use of the endorsed narrative "radius perpendicular to a chord bisects it at the midpoint." Gavilán-Izquierdo and Gallego-Sánchez (2021), in their case study, classify this endorsed narrative as a "verification of sufficient conditions routine". PMT2 also used a theorem he learned in secondary school to determine the value of m, which suggests he could teach learners to integrate various mathematical topics, a crucial skill for them to acquire.

PMT9's response to the problem involved the use of an endorsed narrative similar to that used by PMT2, as well as a substantiation routine of algorithms which helped him to recall the endorsed narrative that "a product of the gradients of two perpendicular line segments is -1". PMT9's calculation of the abscissa of the center of the circle is depicted in **Figure 19**, where he utilized both geometrical and algebraic concepts. The use of recall routines in mathematics is significant since it helps to connect mathematical concepts and build upon previous knowledge (Gavilán-Izquierdo & Gallego-Sánchez, 2021).

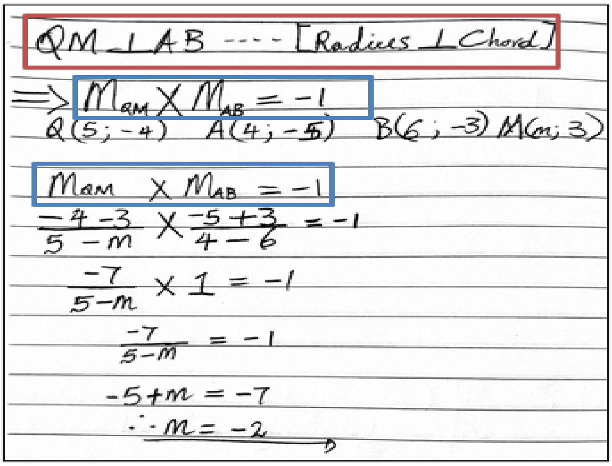


Figure 19. PMT9's solution to number 1 of part A (endorsed narrative [red squared] & routine [blue squared]) (Source: Authors' own elaboration)

Individualized Communication and Mathematical Discourse in Thinking

All of the participants in the study reported engaging in self-discourse, which is a fundamental aspect of commognition. This indicates that they talked to themselves while working on the individual task. They shared their internal thought processes in the following manner:

> I saw that the question asked us to calculate the value. So, if Q is the midpoint–then I thought about what formulas could we use when we calculate the midpoint … then I thought about the midpoint formula, so I said to myself: I can use that to calculate the unknown values for A and Q. Then for the next one, the first thing I thought was the distance formula, I said…because we're given the radius of 10 units, I decided we can use the coordinates of A and the midpoint to find the unknown 'm'. Afterward, I plotted everything on

> the Cartesian plane to confirm my answers (PMT1).

> Well, it's to ask yourself questions like: 'what are the given angles and lengths?' and 'how can I use that information to answer the questions that are being asked?' Then if that is not enough I ask myself more questions and try to identify key information and make some drawings to see what it would look like (PMT4).

> … after getting the first part about figuring out the diagram, then I enjoy the rest of the problem … in number1, I asked myself questions like: 'where will the center of this circle be located on the Cartesian–which quadrant?' Then I marked the possible location, then did the actual calculation, then went back to the diagram and asked myself again if my solution made sense … Then I did the same back and forth–talking to myself … (PMT7).

The participants engaged in self-discursive activity during problem-solving, which helped them identify the key information necessary for solving the problems. The use of symbolic visual mediators was crucial to this process. Their narratives suggest that mathematical problem-solving requires engaging in a process of individualized mathematical discourses that involve moving from a personal discourse to an objectified, impersonal discourse about relations with the object. This process was iterative, as they went back and forth verifying their methods of calculation. This finding is consonant with Ripardo's (2017) proposal on the process of solving mathematical problems involving iteration.

Word Use, Endorsed Narratives, & Routines from Trigonometry Question

The mathematical words "horizontal plane" and "elevation" are used in the statement of the second problem to aid the reader's understanding of what they are required to visualize to solve the second problem. These two mathematical word uses, which appear in everyday language as well as in specialized fields, including mathematics, were critical to participants' ability to develop visual mediators and their individualized mathematical discourses. Participants needed to comprehend these two terms to be able to visualize the entire problem statement and to determine that the two terms were used within the 3D trigonometric context.

The fact that eight participants answered the problem incorrectly could be attributed to their lack of a thorough understanding of these mathematical words. Ho et al. (2019) suggest that individuals engaging in mathematical problem-solving rely on the meanings of standard mathematical terms. Sfard (2008) also emphasizes the importance of "word use", noting that it is essential because it affects what the user can perceive and say about the world. Nevertheless, the participants demonstrated some understanding of how to use the word "elevation" in their visual mediators, as was evident when they described their thought processes. Two reported their approaches, as follows:

> … I tried to visualize that Q was on the same horizontal plane as the angle of elevation. So, I tried my best to visualize what the diagram would possibly look like … (PMT1).

> … then I labelled that vertex Q at the bottom with 90°, to be able to use the trig ratios. And I also placed the angles of elevation accordingly … (PMT10).

PMT1 and PMT10 both demonstrated an understanding of the usage of two key words in a problem, which guided their visualization of the solution. PMT1's iconic visual mediator (**Figure 16**) aligned with the problem statement; PMT10's sketch (**Figure 20**) included both concepts, despite an error in locating the foot of the tower.

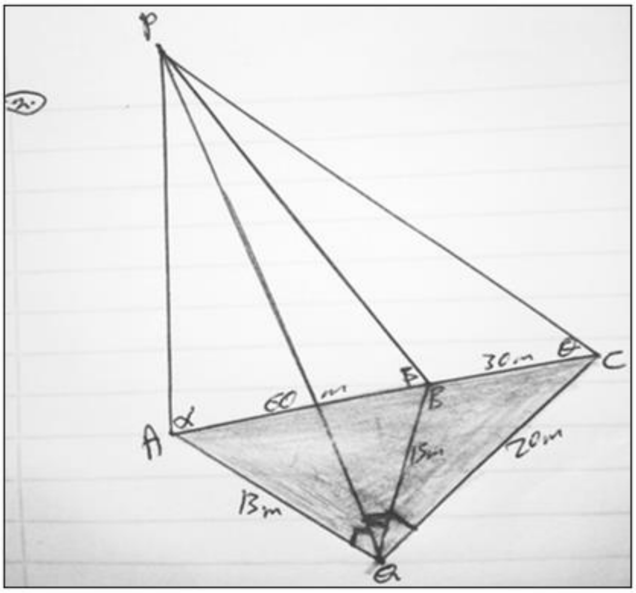


Figure 20. PMT10's visual mediator to question 2 of part A (Source: Authors' own elaboration)

Some participants in the study had difficulty visualizing the second problem due to challenges with representing the mathematical concepts of "angle of elevation" and "horizontal plane" in their iconic visual mediators. The main challenge was visualizing the 3D horizontal plane and the vertical tower in a way that formed the angles of elevation. PMT5 and PMT9 discussed this issue in their reflections:

> … mine was not correct for part A of number 2, because I thought those points–A, B, C, Q–were collinear–like on the straight line–and I thought I was dealing with a 2D problem–hence my sketches [**Figure 13**]. It did not click to me that it

was a 3D shape they were talking about ... Those concepts of angles of elevation and horizontal plane were a bit tricky to sketch (PMT5).

For part A of number 2: … I was not sure if I was dealing with a 2D or 3D. Then I realized later that it was 3D because of the keyword 'horizontal plane' and 'three angles of elevation' and 'vertical tower'. But on my solution, I have a 2D diagram, which is wrong … the terminology they were using needed to be sketched somewhere to make sense of it, otherwise it was going to be very hard to internally visualize what needed to be calculated (PMT9).

Participants were familiar with the necessary mathematical words uses for solving the problem in part A but struggled to visualize the concepts as useful visual mediators. This highlights the importance of a deep understanding of mathematical word uses for successful problem-solving dependent on successful visualization.

PMT1 used the words "opposite" and "adjacent" in his self-discourse while solving the second problem to demonstrate his understanding of the trigonometric definition of tangent. He emphasized the importance of visualizing these terms by sketching them on a piece of paper. This is in line with Rellensmann et al.'s (2017) study, which found students create drawings during mathematical problem-solving to increase their chances of success. PMT1 explained his process, as follows:

So, I saw that was opposite and adjacent sides, so I put $1k$ for the opposite side, and then I put $13k$ for the adjacent. So, I tried to just label everything in pencil and then I tackled the question (PMT1).

He used the arbitrary variable "k" to solve the second problem after understanding the trigonometric definition of tangent in relation to his iconic visual mediator. On the other hand, PMT9 misinterpreted the problem, but correctly used the arbitrary variable x he assigned for the unknown value. However, his incorrect iconic visual mediator led to the incorrect solution. He stated:

I started off with a drawing, and I realized that we were given ratios–like that $tan\alpha = \frac{1}{13}$. Then I said, 'let me assign an arbitrary variable x for those ratios to have something like $tan\alpha = \frac{1x}{13x}$, … Then to solve for x, which I treated as a vertical height of the tower, I used triangle CQA–the big one–to apply the cosine rule (PMT9).

The participant's self-discourse included the use of everyday language words with specific mathematical meanings, such as "vertical height", which has a different connotation in mathematics than in common language. The participant also demonstrated recall routine of the "cosine rule", which is a key rule taught in the South African secondary school curriculum and was important for successfully solving the second problem. Sfard (2007) emphasizes the importance of recall routines in developing discursive fluency in mathematical activities.

Validating Comparisons of Both Parts of Performance Tests

All participants were requested to compare the difficulty of part A and part B during the interview sessions to determine whether the accompanying iconic visual mediators enabled them to visualize the problems more clearly. The majority of participants agreed that part B, which included the supporting iconic visual mediators, was simpler. PMT4 and PMT7 commented:

The one that was easy and enjoyable between the two was the one with diagrams, and that will be part B … I am saying that because if you visualize on your own and try to come up with the diagram from the question trying to understand that English … so visualizing the word problem and trying to come up with that correct diagram is challenging (PMT4).

For part A, I could not come up with a correct diagram at first until very late … and still after that, I could not work it out and I decided to skip it (PMT7).

This suggests that the use of iconic visual mediators in test questions can assist comprehension, resulting in more successful problem-solving, especially when language is a barrier. The accuracy of these visual mediators can minimize errors and make it easier to understand the problem. In part B of the test, where iconic visual mediators were provided, students were able to arrive at different solutions to a more challenging problem compared to part A, where no iconic visual mediators were used. Supardi et al. (2021) also established that using visual mediators in the form of images can aid students in solving mathematical problems, thus reducing the likelihood of errors. PMT9 and PMT6 were able to solve the second problem, which they had found more challenging, and arrived at different solutions than they had for the same problem in part A. PMT9 explained:

For number 2 of parts A and B my answers are not the same. I think the correct answer is from part B because there I used the correct diagram, but for part A my diagram was not correct, then my answer is not correct there … so the correct answer is supposed to be 4.3 m, and not 70 m (PMT9).

Figure 21 depicts PMT9's responses to the second problem in parts A and B. Although the iconic visual

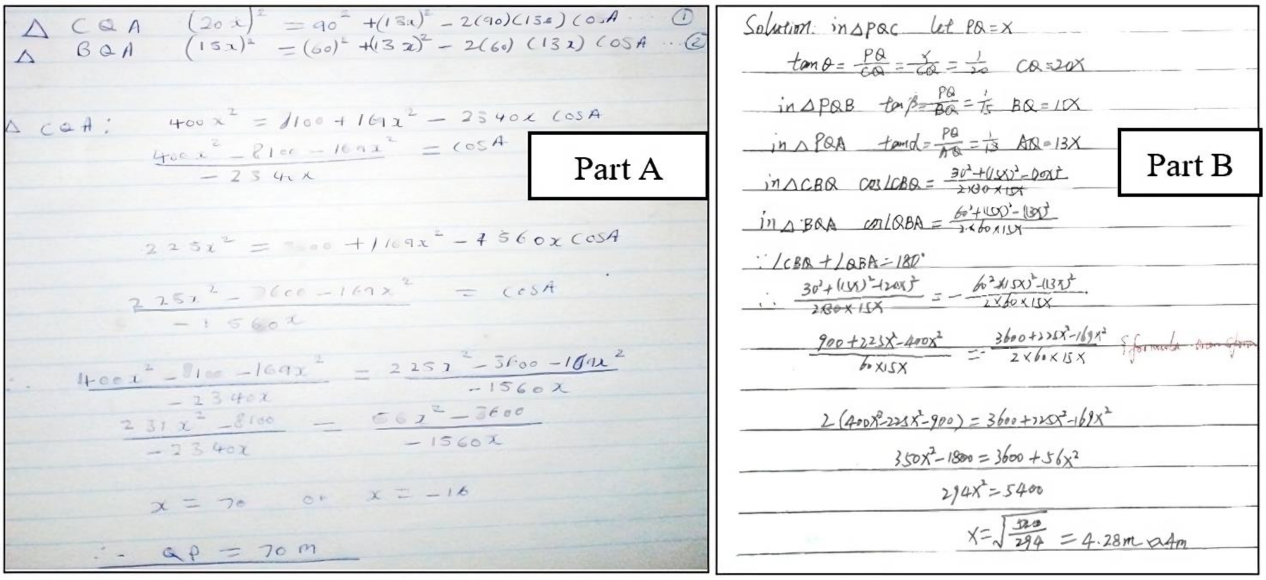


Figure 21. PMT9's solutions to question 2 in part A (left) & part B (right) (Source: Authors' own elaboration)

mediator was helpful in visualizing the problem, eight of 10 participants continued to experience difficulty arriving at the correct solution for the second problem in part B. PMT3 shared his experience:

> Part B was much better for me because … the key to having a correct diagram to work with, make it much easier to visualize things even further. Even though I am still not getting to the final answer for the trig problem, at least now I have more clue as to what could work, though I'm not sure for now (PMT3).

Figure 22 validates PMT3's account of the difficulty he encountered in solving the second problem. **Figure 22** shows PMT3's initial and subsequent attempts, which failed to yield the correct solution–even in part B, where the appropriate iconic visual mediator was provided.

PMT3's experience was similar to those of PMT5 and PMT7, described, as follows:

> Number 1 of both tasks was easy for me. I just had challenges with both number 2s from part A and B. I still do not know how to do it (PMT5).

> Part B eliminated all those challenges since everything was already drawn … but still I could not work it out and I decided to skip it. I think what challenged me the most is that I have not worked with these types of problems for a long time (PMT7).

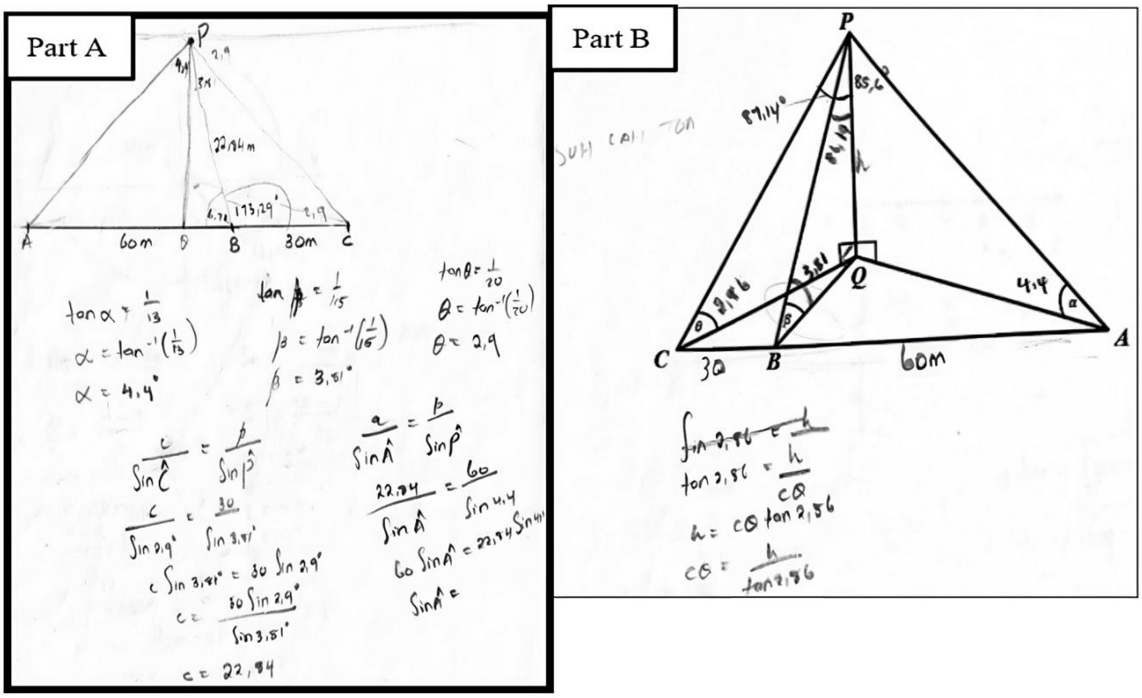


Figure 22. PMT3's solutions to question 2 in part A (left) & part B (right) (Source: Authors' own elaboration)

The difficulties encountered by the participants in solving performance test problems were not due solely to their visualization skills, as evidenced by their comments and accompanying visuals. One participant, PMT7, was unable to solve a problem despite the aid of an iconic visual mediator because she was not familiar with that type of problem. This finding is consistent with previous research by Supardi et al. (2021) that shows students may still make mistakes despite using correct visual mediators. Another study by Rellensmann et al. (2017) found that using appropriate visual mediators does not guarantee solution accuracy. The participants' challenges may also be attributed to not knowing the appropriate problem-solving strategy. However, as the study did not analyze participants' responses for the purpose of determining whether they were correct or incorrect, their errors and challenges were not further explored.

CONCLUSIONS

Visualization techniques and discursive properties in problem-solving play a key role in the teaching and learning of mathematics. In this study, PMTs were found to rely on mental visualization for simple problems but required both iconic and symbolic visual mediators for more complex problems. Language was found to play a crucial role in self-discursive activity, which contributes to successful visualization. The study established the significance of discursive properties such as word uses, endorsed narratives, routines, and visual mediators in the problem-solving process.

The findings of this study could be used to inform mathematics teacher education and assist in the development of effective pedagogical strategies for teaching problem-solving skills. By observing and analyzing the ways in which PMTs visually represent mathematical concepts and how they communicate their thinking during problem-solving, instructors can identify areas of strength and weakness in their pedagogical skills. Additionally, by assisting PMTs develop their visualization and discourse abilities, instructors can better prepare them to teach mathematics in ways that are engaging and accessible to diverse groups of learners. This can lead to more effective teaching and improved student learning outcomes in mathematics.

Nevertheless, this study used a qualitative research approach, which may limit the generalizability and objectivity of the findings compared to quantitative research methods. A large-scale mixed research study focusing on the visualization techniques and mathematical discourses used by PMTs during mathematical problem-solving, in relation to their teaching approaches, is recommended for future research.

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